

J-CLASS ABELIAN SEMIGROUPS OF MATRICES ON \mathbb{C}^n AND HYPERCYCLICITY

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ABSTRACT. We give a characterization of hypercyclic finitely generated abelian semigroups of matrices on \mathbb{C}^n using the extended limit sets (the J-sets). Moreover we construct for any $n \geq 2$ an abelian semigroup G of $\text{GL}(n, \mathbb{C})$ generated by $n + 1$ diagonal matrices which is locally hypercyclic but not hypercyclic and such that $J_G(e_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$, where (e_1, \dots, e_n) is the canonical basis of \mathbb{C}^n . This gives a negative answer to a question raised by Costakis and Manoussos.

1. Introduction

Let $M_n(\mathbb{C})$ be the set of all square matrices over \mathbb{C} of order $n \geq 1$ and by $\text{GL}(n, \mathbb{C})$ the group of invertible matrices of $M_n(\mathbb{C})$. Let G be a finitely generated abelian sub-semigroup of $M_n(\mathbb{C})$. For a vector $v \in \mathbb{C}^n$, we consider the orbit of G through v : $G(v) = \{Av : A \in G\} \subset \mathbb{C}^n$. A subset $E \subset \mathbb{C}^n$ is called *G-invariant* if $A(E) \subset E$ for any $A \in G$. The orbit $G(v) \subset \mathbb{C}^n$ is called *dense* in \mathbb{C}^n if $\overline{G(v)} = \mathbb{C}^n$, where \overline{E} denotes the closure of a subset $E \subset \mathbb{C}^n$. The semigroup G is called *hypercyclic* if there exists a vector $v \in \mathbb{C}^n$ such that $G(v)$ is dense in \mathbb{C}^n . Hypercyclic is also called topologically transitive. We refer the reader to the recent book [3] and [7] for a thorough account on hypercyclicity. In [5], Costakis and Manoussos introduced the concept of extended limit set to G : Suppose that G is generated by p matrices A_1, \dots, A_p ($p \geq 1$) then for $x \in \mathbb{C}^n$, we define the extended limit set $J_G(x)$ of x under G to be the set of $y \in \mathbb{C}^n$ for which there exists a sequence $(x_m)_m \subset \mathbb{C}^n$ and sequences of non-negative integers $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ with

$$(1.1) \quad k_m^{(1)} + k_m^{(2)} + \dots + k_m^{(p)} \rightarrow +\infty$$

such that $x_m \rightarrow x$ and $A_1^{k_m^{(1)}} A_2^{k_m^{(2)}} \dots A_p^{k_m^{(p)}} x_m \rightarrow y$.

Note that condition (1.1) is equivalent to having at least one of the sequences $\{k_m^{(j)} : m \in \mathbb{N}\}$ for $j = 1, 2, \dots, p$ containing a strictly increasing

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subsequence tending to $+\infty$. We say that G is *locally hypercyclic* if there exists a vector $v \in \mathbb{C}^n \setminus \{0\}$ such that $J_G(v) = \mathbb{C}^n$. This notion is a “localization” of the concept of hypercyclicity, this can be justified by the following: $J_G(x) = \mathbb{C}^n$ if and only if for every open neighborhood $U_x \subset \mathbb{C}^n$ of x and every nonempty open set $V \subset \mathbb{C}^n$ there exists $A \in G$ such that $A(U_x) \cap V \neq \emptyset$.

In \mathbb{C}^n , no matrice can be locally hypercyclic (see [4]). However, what is rather remarkable is that in \mathbb{C}^n or \mathbb{R}^n , a pair of commuting matrices exists which forms a locally hypercyclic, non-hypercyclic semigroup.

The main purpose of this paper is twofold: firstly, we give a characterization of hypercyclic finitely generated abelian semigroup of $M_n(\mathbb{C})$ through the use of the extended limit sets. We show that G is hypercyclic if and only if there exists a vector v in an open set V , defined according to the structure of G , such that $J_G(v) = \mathbb{C}^n$ (Theorem 1.2). Secondly, we answer negatively (Theorem 1.5) the following question raised by Costakis and Manoussos in [5]: is it true that a locally hypercyclic abelian semigroup G generated by matrices A_1, \dots, A_p is hypercyclic whenever $J_G(u_k) = \mathbb{C}^n$, $k = 1, \dots, n$, for a basis (u_1, \dots, u_n) of \mathbb{C}^n ? However, we prove that the question is true (see Proposition 1.6) for any abelian semigroup G consisting of lower triangular matrices on \mathbb{C}^n with all diagonal elements equal.

Before stating our main results, let introduce the following notations and definitions. Denote by:

- $\mathcal{B}_0 = (e_1, \dots, e_n)$ the canonical basis of \mathbb{C}^n .
- $\mathbb{N}_0 = \mathbb{N} \setminus \{0\}$.
- I_n the identity matrix on \mathbb{C}^n .

Let $n \in \mathbb{N}_0$ fixed. For each $m = 1, 2, \dots, n$, denote by:

- $\mathbb{T}_m(\mathbb{C})$ the set of matrices over \mathbb{C} of the form:

$$\begin{bmatrix} \mu & & & 0 \\ a_{2,1} & \ddots & & \\ \vdots & \ddots & \ddots & \\ a_{m,1} & \dots & a_{m,m-1} & \mu \end{bmatrix} \quad (1)$$

- $\mathbb{T}_m^*(\mathbb{C}) = \mathbb{T}_m(\mathbb{C}) \cap \text{GL}(m, \mathbb{C})$ the group of matrices of the form (1) with $\mu \neq 0$.

Let $r \in \mathbb{N}$ and $\eta = (n_1, \dots, n_r)$ be a sequence of positive integers such that $n_1 + \dots + n_r = n$. In particular, $r \leq n$.

Denote by

- $\mathcal{K}_{\eta,r}(\mathbb{C}) := \mathbb{T}_{n_1}(\mathbb{C}) \oplus \dots \oplus \mathbb{T}_{n_r}(\mathbb{C})$.
- $\mathcal{K}_{\eta,r}^*(\mathbb{C}) := \mathcal{K}_{\eta,r}(\mathbb{C}) \cap \text{GL}(n, \mathbb{C})$.
- $U := \prod_{k=1}^r (\mathbb{C}^* \times \mathbb{C}^{n_k-1})$.
- v^T the transpose of a vector $v \in \mathbb{C}^n$.

- $u_0 = [e_{1,1}, \dots, e_{r,1}]^T \in \mathbb{C}^n$ where $e_{k,1} = [1, 0, \dots, 0]^T \in \mathbb{C}^{n_k}$, $k = 1, \dots, r$.

Given any abelian sub-semigroup G of $M_n(\mathbb{C})$, we introduce the triangular representation for G .

Proposition 1.1. ([2], Proposition 2.2) *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$. Then there exists a $P \in GL(n, \mathbb{C})$ such that $PGP^{-1} \subset \mathcal{K}_{\eta,r}(\mathbb{C})$, for some $\eta \in (\mathbb{N}_0)^r$ and $r \in \{1, \dots, n\}$.*

This reduces the existence of a dense orbit to a question concerning sub-semigroups of $\mathcal{K}_{\eta,r}(\mathbb{C})$. For such a choice of matrix P , we let:

- $v_0 = Pu_0$.
- $V = P(U)$, it is a dense open set in \mathbb{C}^n .

Our principal results are the following:

Theorem 1.2. *Let G be a finitely generated abelian semigroup of matrices on \mathbb{C}^n . If $J_G(v) = \mathbb{C}^n$ for some $v \in V$ then $G(v) = \mathbb{C}^n$.*

Corollary 1.3. *Under the hypothesis of Theorem 1.2, the following are equivalent:*

- (i) G is hypercyclic.
- (ii) $J_G(v_0) = \mathbb{C}^n$.
- (iii) $\overline{G(v_0)} = \mathbb{C}^n$.

Corollary 1.4. *Under the hypothesis of Theorem 1.2, if G is not hypercyclic then $E := \{x \in \mathbb{C}^n : J_G(x) = \mathbb{C}^n\} \subset \bigcup_{k=1}^r H_k$, ($r \leq n$) where H_k are G -invariant vector subspaces of \mathbb{C}^n with dimension $n - 1$.*

Remark. In the case $n = 1$, we have $V = \mathbb{C}^*$ and by Theorem 1.2, we conclude that a sub-semigroup G of \mathbb{C} is hypercyclic if and only if it is locally hypercyclic.

Theorem 1.5. *Let $n \geq 2$ be an integer. Then there exists an abelian semigroup G generated by diagonal matrices $A_1, \dots, A_{n+1} \in GL(n, \mathbb{C})$ which is not hypercyclic such that $J_G(e_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$.*

Proposition 1.6. *Let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{C})$. If there exists a basis (e'_1, \dots, e'_n) of \mathbb{C}^n such that $J_G(e'_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$, then G is hypercyclic.*

2. Preliminaries and basic notions

Let G be a sub-semigroup of $\mathbb{T}_n(\mathbb{C})$. Every element $B \in G$ is written in the form

$$B = \begin{bmatrix} B^{(1)} & 0 \\ L_B & \mu_B \end{bmatrix}$$

with $B^{(1)} \in \mathbb{T}_{n-1}(\mathbb{C})$, $L_B \in M_{1,n-1}(\mathbb{C})$ and $\mu_B \in \mathbb{C}$.

Denote by

- $G^{(1)} = \{B^{(1)} : B \in G\}$.
- $F_G = \text{vect}(\{(B - \mu_B I_n)e_i \in \mathbb{C}^n : 1 \leq i \leq n-1, B \in G\})$ the vector subspace generated by the family of vectors $\{(B - \mu_B I_n)e_i \in \mathbb{C}^n : 1 \leq i \leq n-1, B \in G\}$.
- $\text{rank}(F_G)$ the rank of F_G . We have $\text{rank}(F_G) \leq n-1$.
- For every $x = [x_1, \dots, x_n]^T \in \mathbb{C}^n$, we let $x^{(1)} = [x_1, \dots, x_{n-1}]^T \in \mathbb{C}^{n-1}$. We have $x = [x^{(1)}, x_n]^T$ and $e_k = [e_k^{(1)}, 0]^T$, $k = 1, \dots, n-2$.
- $F_G^{(1)} = \text{vect}(\{(B^{(1)} - \mu_B I_{n-1})e_k^{(1)} : 1 \leq k \leq n-2, B \in G\})$.

For every $B \in \mathcal{K}_{n,r}(\mathbb{C})$, write $B = \text{diag}(B_1, \dots, B_r)$ where $B_k \in \mathbb{T}_{n_k}(\mathbb{C})$.

If G is an abelian sub-semigroup of $\mathcal{K}_{n,r}(\mathbb{C})$, denote by:

$G_k = \{B_k : B \in G\}$, it is an abelian sub-semigroup of $\mathbb{T}_{n_k}(\mathbb{C})$.

For every $x = [x_1, \dots, x_r]^T \in \mathbb{C}^n$ where $x_k = [x_{k,1}, \dots, x_{k,n_k}]^T \in \mathbb{C}^{n_k}$, we let:

- $H_{x_k} = \mathbb{C}x_k + F_{G_k}$, $k = 1, \dots, r$.
- $H_x = \bigoplus_{k=1}^r H_{x_k}$.

We start with the following lemmas:

Lemma 2.1. *Let G be an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. Under the notations above, for every $x \in \mathbb{C}^n$, H_x is G -invariant.*

Proof. It suffices to prove that H_{x_k} is G_k -invariant: write $x = x_k$ and $G = G_k$. In this case $H_x = \mathbb{C}x + F_G$. Let $w = [w_1, \dots, w_n]^T \in H_x$ and $B \in G$ with eigenvalue μ . We have $Bw = \mu w + (B - \mu I_n)w = \mu w + \sum_{i=1}^{n-1} w_k (B - \mu I_n)e_i$. Since $w, (B - \mu I_n)e_i \in H_x$ and H_x is a vector space, we have $Bw \in H_x$. \square

Proposition 2.2. *Let G be an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$ generated by A_1, \dots, A_p . If $J_G(u) = \mathbb{C}^n$ for some $u \in U$ then for every $k = 1, \dots, r$, $\text{rank}(F_{G_k}) = n_k - 1$.*

Proof. Write $u = [u_1, \dots, u_r]^T$, $u_k \in \mathbb{C}^* \times \mathbb{C}^{n_k-1}$, $A_j = \text{diag}(A_{j,1}, \dots, A_{j,r})$, $A_{j,k} \in \mathbb{T}_{n_k}$, $j = 1, \dots, p$; $k = 1, \dots, r$. First, we will show that $J_{G_k}(u_k) = \mathbb{C}^{n_k}$. For this, let $x_k \in \mathbb{C}^{n_k}$ and $y = [y_1, \dots, y_r]^T \in \mathbb{C}^n$ such that $y_i = 0 \in \mathbb{C}^{n_i}$ if $i \neq k$ and $y_k = x_k$. As $J_G(u) = \mathbb{C}^n$, there exist two sequences $(x_m)_m \subset \mathbb{C}^n$ and $(B_m)_m \subset G$ such that

$$\lim_{m \rightarrow +\infty} x_m = u \quad \text{and} \quad \lim_{m \rightarrow +\infty} B_m x_m = y. \quad (1)$$

Write $x_m = [x_{m,1}, \dots, x_{m,r}]^T$, $x_{m,k} \in \mathbb{C}^{n_k}$ and $B_m = \text{diag}(B_{m,1}, \dots, B_{m,r})$, $B_{m,k} \in \mathbb{T}_{n_k}(\mathbb{C})$, $k = 1, \dots, r$. By (1), we have

$$\lim_{m \rightarrow +\infty} x_{m,k} = u_k \quad \text{and} \quad \lim_{m \rightarrow +\infty} B_{m,k} x_{m,k} = y_k = x_k.$$

Therefore $x_k \in J_{G_k}(u_k)$. It follows that $J_{G_k}(u_k) = \mathbb{C}^{n_k}$.

Second, one can then suppose that $G \subset \mathbb{T}_n(\mathbb{C})$ and $u \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. It is clear that $u \notin F_G$. By Lemma 2.1, $H_u = \mathbb{C}u + F_G$ is G -invariant. Assume that $\mathbb{C}^n \setminus H_u \neq \emptyset$, so let $y \in \mathbb{C}^n \setminus H_u$. Then there exist two sequences $(x_m)_m \subset \mathbb{C}^n$ and $(B_m)_m \subset G$ such that $\lim_{m \rightarrow +\infty} x_m = u$ and $\lim_{m \rightarrow +\infty} B_m x_m = y$. Let $H_{x_m} = \mathbb{C}x_m + F_G$ for every $m \in \mathbb{N}$. By Lemma 2.1, H_{x_m} is G -invariant, so $B_m x_m \in H_{x_m}$, for every $m \in \mathbb{N}$. Write $B_m x_m = \alpha_m x_m + z_m$, $\alpha_m \in \mathbb{C}$ and $z_m \in F_G$. We distinguish two cases:

- If $(\alpha_m)_m$ is bounded, one can suppose by passing to a subsequence, that $(\alpha_m)_{m \geq 1}$ is convergent, say $\lim_{m \rightarrow +\infty} \alpha_m = a \in \mathbb{C}$. It follows that $\lim_{m \rightarrow +\infty} z_m = y - au \in F_G$ and so $y \in H_u$, a contradiction.
- If $(\alpha_m)_m$ is not bounded, one can suppose by passing to a subsequence, that $\lim_{m \rightarrow +\infty} |\alpha_m| = +\infty$, then $\lim_{m \rightarrow +\infty} \frac{1}{\alpha_m} z_m = -u \in F_G$, a contradiction. We conclude that $H_u = \mathbb{C}^n$ and so $\dim(F_G) = n - 1$. \square

3. Proof of Theorem 1.2, Corollaries 1.3 and 1.4

Let recall the following results.

Proposition 3.1. ([1], Proposition 5.1) *Let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{C})$, $n \geq 1$. If $\text{rank}(F_G) = n - 1$, then there exists an injective linear map $\varphi : \mathbb{C}^n \rightarrow \mathbb{T}_n(\mathbb{C})$ such that:*

- (i) *for every $v \in \mathbb{C}^n$, $\varphi(v)e_1 = v$*
- (ii) *$G \subset \varphi(\mathbb{C}^n)$.*

The map φ in Proposition 3.1 can be precise, from the proof of ([1], Proposition 5.1), as follows:

Corollary 3.2. *Under the hypothesis of Proposition 3.1, and for $n \geq 2$, there exists a linear map $\eta : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-2}$ such that for every*

$$v = [v_1, \dots, v_n]^T \in \mathbb{C}^n, \varphi(v) = \begin{bmatrix} \varphi^{(1)}(v^{(1)}) & 0 \\ [v_n, (\eta(v^{(1)}))^T]^T & v_1 \end{bmatrix}, \text{ with}$$

$$v^{(1)} = [v_1, \dots, v_{n-1}]^T \in \mathbb{C}^n \text{ and } \varphi^{(1)} : \mathbb{C}^{n-1} \rightarrow \mathbb{T}_{n-1}(\mathbb{C}) \text{ is the injective linear map associated to } G^{(1)} \text{ given by Proposition 3.1.}$$

Lemma 3.3. *Let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{C})$, $n \geq 1$. Suppose that $\text{rank}(F_G) = n - 1$. Let $u, v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$, $(u_m)_{m \in \mathbb{N}}$ in $\mathbb{C}^* \times \mathbb{C}^{n-1}$ and $(B_m)_{m \in \mathbb{N}}$ in G such that $\lim_{m \rightarrow +\infty} u_m = u$ and $\lim_{m \rightarrow +\infty} B_m u_m = v$. If $(B_m^{(1)})_{m \in \mathbb{N}}$ is bounded then $(B_m)_{m \in \mathbb{N}}$ is bounded.*

Proof. By Proposition 3.1, there exists an injective linear map $\varphi : \mathbb{C}^n \rightarrow \mathbb{T}_n(\mathbb{C})$ such that $G \subset \varphi(\mathbb{C}^n)$ and $\varphi(w)e_1 = w$ for every $w \in \mathbb{C}^n$.

Write $B_m = \varphi(w_m)$, $w_m = [z_{m,1}, \dots, z_{m,n}]^T \in \mathbb{C}^n$. By Corollary 3.2, we have

$$B_m = \varphi(w_m) = \begin{bmatrix} \varphi^{(1)}(w_m^{(1)}) & 0 \\ [z_{m,n}, (\eta(w_m^{(1)}))^T]^T & z_{m,1} \end{bmatrix},$$

where $\varphi^{(1)}$ is the injective linear map associated to $G^{(1)}$ and $\eta : \mathbb{C}^{n-1} \rightarrow \mathbb{C}^{n-2}$ is a linear map. Then $w_m^{(1)} = \varphi^{(1)}(w_m^{(1)})u_0^{(1)} = B_m^{(1)}u_0^{(1)}$ and thus $(w_m^{(1)})_{m \geq 1}$ is bounded. So $(\eta(w_m^{(1)}))_{m \geq 1}$ and $(z_{m,1})_{m \geq 1}$ are also bounded. Write $u_m = [u_{m,1}, \dots, u_{m,n}]^T \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. Then we have $\lim_{m \rightarrow +\infty} u_{m,1} = u_1 \neq 0$ and $\lim_{m \rightarrow +\infty} z_{m,n}u_{m,1} = v_1 \neq 0$. Thus $\lim_{m \rightarrow +\infty} z_{m,n} = \frac{v_1}{u_1}$ and so $(z_{m,n})_{m \geq 1}$ is bounded. We deduce that $(B_m)_{m \in \mathbb{N}}$ is bounded, this completes the proof. \square

Lemma 3.4. *Let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{C})$ such that $\text{rank}(F_G) = n - 1$. Let $u, v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. If two sequences $(u_m)_{m \in \mathbb{N}}$ in $\mathbb{C}^* \times \mathbb{C}^{n-1}$ and $(B_m)_{m \in \mathbb{N}}$ in G such that $\lim_{m \rightarrow +\infty} u_m = u$ and $\lim_{m \rightarrow +\infty} B_m u_m = v$ then $(B_m)_{m \in \mathbb{N}}$ is bounded.*

Proof. The proof is done by induction on n . For $n = 1$, we have $u_m \in \mathbb{C}^*$, $B_m = \lambda_m \in \mathbb{C}$ and $u, v \in \mathbb{C}^*$. The conditions $\lim_{m \rightarrow +\infty} u_m = u$ and $\lim_{m \rightarrow +\infty} B_m u_m = v$ show that $\lim_{m \rightarrow +\infty} B_m = \frac{v}{u}$, hence $(B_m)_{m \in \mathbb{N}}$ is bounded. Suppose the Lemma is true up to dimension $n - 1$ and let G be an abelian sub-semigroup of $\mathbb{T}_n(\mathbb{C})$. Let $u, v \in \mathbb{C}^* \times \mathbb{C}^{n-1}$, $(u_m)_{m \in \mathbb{N}}$ a sequence in $\mathbb{C}^* \times \mathbb{C}^{n-1}$ and $(B_m)_{m \in \mathbb{N}}$ a sequence in G such that $\lim_{m \rightarrow +\infty} u_m = u$ and

$\lim_{m \rightarrow +\infty} B_m u_m = v$. We let $u = [u_1, \dots, u_n]^T$, $v = [v_1, \dots, v_n]^T$ and $u_m = [u_{m,1}, \dots, u_{m,n}]^T \in \mathbb{C}^n$. We have $\lim_{m \rightarrow +\infty} u_m^{(1)} = u^{(1)}$ and $\lim_{m \rightarrow +\infty} B_m^{(1)} u_m^{(1)} = v^{(1)}$. The set $G^{(1)}$ is an abelian sub-semigroup of $\mathbb{T}_{n-1}(\mathbb{C})$ and $\text{rank}(F_{G^{(1)}}) = n - 2$. By induction hypothesis applied to $G^{(1)}$ on \mathbb{C}^{n-1} , the sequence $(B_m^{(1)})_{m \in \mathbb{N}}$ is bounded. Therefore, by Lemma 3.3, $(B_m)_m$ is bounded. \square

Corollary 3.5. *Let G be an abelian sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$ generated by A_1, \dots, A_p , $p \geq 1$. Suppose that $\text{rank}(F_{G_k}) = n_k - 1$, $k = 1, \dots, r$. If $x, y \in U$ and two sequences $(B_m)_m \subset G$ and $(x_m)_m \subset \mathbb{C}^m$ such that $\lim_{m \rightarrow +\infty} x_m = x$ and $\lim_{m \rightarrow +\infty} B_m x_m = y$ then $(B_m)_m$ is bounded.*

Proposition 3.6. [5] *Let G be an abelian sub-semigroup of $M_n(\mathbb{C})$ generated by A_1, \dots, A_p , $p \geq 1$. Then G is hypercyclic if and only if $J_G(x) = \mathbb{C}^n$ for every $x \in \mathbb{C}^n$.*

Proof of Theorem 1.2. One can assume, by Proposition 1.1, that G is a sub-semigroup of $\mathcal{K}_{\eta,r}(\mathbb{C})$. Suppose that $J_G(u) = \mathbb{C}^n$ where $u \in U$. Then by Proposition 2.2, $\text{rank}(F_{G_k}) = n_k - 1$, for every $k = 1, \dots, r$. Let $y \in U$, then there exist two sequences $(B_m)_m \subset G$ and $(x_m)_m \subset \mathbb{C}^n$ satisfying:

$$\lim_{m \rightarrow +\infty} x_m = u \quad \text{and} \quad \lim_{m \rightarrow +\infty} B_m x_m = y.$$

So by Corollary 3.5, $(B_m)_{m \geq 1}$ is bounded: $\|B_m\| \leq M$ for some $M > 0$ where $\|\cdot\|$ is the Euclidean norm on \mathbb{C}^n . Then

$$\begin{aligned} \|B_m u - y\| &= \|B_m u - B_m x_m + B_m x_m - y\| \\ &\leq \|B_m u - B_m x_m\| + \|B_m x_m - y\| \\ &\leq \|B_m\| \|u - x_m\| + \|B_m x_m - y\| \\ &\leq M \|u - x_m\| + \|B_m x_m - y\| \end{aligned}$$

Thus $\lim_{m \rightarrow +\infty} B_m u = y$ and so $y \in \overline{G(u)}$. It follows that $U \subset \overline{G(u)}$ and as $\overline{U} = \mathbb{C}^n$, we get $\overline{G(u)} = \mathbb{C}^n$. \square

Proof of Corollary 1.3. (i) \implies (ii) follows from Proposition 3.6. (ii) \implies (iii): this results from Theorem 1.2. (iii) \implies (i): is clear.

Proof of Corollary 1.4. If G is not hypercyclic then by Theorem 1.2, $J_G(v) \neq \mathbb{C}^n$ for any $v \in V$, thus $V \cap E = \emptyset$ and therefore $E \subset \mathbb{C}^n \setminus V = \bigcup_{k=1}^r H_k$

with $H_k = P\left(\{x = [x_1, \dots, x_r]^T, x_i \in \mathbb{C}^{n_i}, \text{ if } i \neq k, \text{ and } x_k \in \{0\} \times \mathbb{C}^{n_k-1}\}\right)$. \square

4. Proof of Theorem 1.5 and Proposition 1.6

For the proof of Theorem 1.5, we will make use of the following result:

Lemma 4.1. [6]. *If $a \in \mathbb{C}$ with $|a| > 1$, then there is a dense set $\Delta_a \subset \{z \in \mathbb{C}, |z| < 1\}$ such that for any $b \in \Delta_a$, we have that $\{a^k b^l : k, l \in \mathbb{N}\}$ is dense in \mathbb{C} .*

Proof of Theorem 1.5. Let $a \in \mathbb{C}$ with $|a| > 1$. By Lemma 4.1, there exists $b \in \mathbb{C}$ with $\frac{1}{|a|} < |b| < 1$ such that $\{a^k b^l : k, l \in \mathbb{N}\}$ is dense in \mathbb{C} . Consider the abelian sub-semigroup G of $\mathcal{K}_{\eta,r}^*(\mathbb{C})$ generated by B, A_1, \dots, A_n , where

$$B = bI_n \text{ and } A_k = \text{diag}(\underbrace{a, \dots, a}_{(k-1)\text{-terms}}, 1, a, \dots, a), \quad k = 1, \dots, n.$$

• First, we will show that G is not hypercyclic: for this, it is equivalent to prove, by Corollary 1.3, that $\overline{G(u_0)} \neq \mathbb{C}^n$ where $u_0 = [1, \dots, 1]^T$. We have

$$G(u_0) = \left\{ [b^m a^{k_2} \dots a^{k_n}; b^m a^{k_1} a^{k_3} \dots a^{k_n}; \dots; b^m a^{k_1} \dots a^{k_{n-1}}]^T : m, k_1, \dots, k_n \in \mathbb{N} \right\}$$

Observe that for every $x = [x_1, \dots, x_n]^T \in G(u_0)$, we have $\frac{x_i}{x_n} = a^{k_n - k_i}$, $i = 1, \dots, n$. So $G(u_0) \subset F$ where

$$F = \left\{ [a^{\ell_1} \lambda; a^{\ell_2} \lambda; \dots; a^{\ell_n} \lambda]^T : \lambda \in \mathbb{C}, \ell_1, \dots, \ell_n \in \mathbb{Z} \right\}.$$

We set

$$D_k = \text{diag}(\underbrace{1, \dots, 1}_{(k-1)\text{-terms}}, a, 1, \dots, 1), \quad k = 1, \dots, n$$

and

$$\Delta = \{[\lambda, \dots, \lambda]^T : \lambda \in \mathbb{C}\}.$$

Then we have

$$F = \bigcup_{\substack{\ell_i \in \mathbb{Z} \\ 1 \leq i \leq n}} D_1^{\ell_1} \dots D_n^{\ell_n}(\Delta).$$

It is plain that $\overline{F} = F \cup \bigcup_{k=1}^n E_k$, where

$$E_k = \{x = [x_1, \dots, x_n]^T : x_i \in \mathbb{C}, \text{ if } i \neq k \text{ and } x_k = 0\}.$$

Since $\overset{\circ}{E}_k = \emptyset$, it follows that $\bigcup_{k=1}^{\overset{\circ}{n}} E_k = \emptyset$, where $\overset{\circ}{M}$ denotes the interior of a subset $M \subset \mathbb{C}^n$. Moreover, since $\overline{D_1^{k_1} \dots D_n^{k_n}(\Delta)} = \emptyset$, it follows, by Baire's theorem, that $\overline{G(u_0)} = \emptyset$ and so $\overline{G(u_0)} \neq \mathbb{C}^n$.

• Second, we will show that $J_G(e_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$. Fix a vector $y = [y_1, \dots, y_n]^T \in \mathbb{C}^n$. Choose two sequences of positive integers $(i_m)_{m \in \mathbb{N}}$ and $(j_m)_{m \in \mathbb{N}}$ with $i_m, j_m \rightarrow +\infty$ such that $\lim_{m \rightarrow +\infty} a^{i_m} b^{j_m} = y_k$. As $n \geq 2$, one can choose $s \in \{1, \dots, n\}$ such that $s \neq k$. We let $B_m = A_s^{i_m} B^{j_m} A_k^{j_m}$. Then we have $B_m = \text{diag}(a_{m,1}, \dots, a_{m,n})$ where

$$a_{m,l} = \begin{cases} a^{i_m} a^{j_m} b^{j_m} & \text{if } l \neq k, s \\ a^{j_m} b^{j_m} & \text{if } l = s \\ a^{i_m} b^{j_m} & \text{if } l = k \end{cases}$$

Hence, $\lim_{m \rightarrow +\infty} a_{m,k} = y_k$ (3). Moreover, since $\frac{1}{|a|} < |b| < 1$, we have $\lim_{m \rightarrow +\infty} a_{m,l} = +\infty$ for every $l \neq k$ and hence $\lim_{m \rightarrow +\infty} x_m = e_k$. We set $x_m = (x_{m,1}, \dots, x_{m,n})$ where

$$x_{m,l} = \begin{cases} \frac{y_l}{a_{m,l}} & \text{if } l \neq k \\ 1 & \text{if } l = k \end{cases}$$

Then $B_m x_m = (y_1, \dots, y_{k-1}, a_{m,k}, y_{k+1}, \dots, y_n)$, and by (3), it follows that $\lim_{m \rightarrow +\infty} B_m x_m = y$. We conclude that $y \in J_G(e_k)$ and therefore $J_G(e_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$. \square

Proof of Proposition 1.6. Since (e'_1, \dots, e'_n) is a basis of \mathbb{C}^n , there exists $i_0 \in \{1, \dots, n\}$ such that $e'_{i_0} \in \mathbb{C}^* \times \mathbb{C}^{n-1}$. As $V = U = \mathbb{C}^* \times \mathbb{C}^{n-1}$ and $J_G(e'_{i_0}) = \mathbb{C}^n$ then by Theorem 1.2, $\overline{G(e'_{i_0})} = \mathbb{C}^n$ and hence G is hypercyclic. \square

The following questions arose naturally.

Question 1. Find analogous to Theorems 1.2 and 1.5 for the real case?

Question 2. Let $1 \leq r \leq n$ be an integer. Is it true that there exists a finitely generated abelian semigroup G of $\mathcal{K}_{n,r}(\mathbb{C})$ which is not hypercyclic such that $J_G(e_k) = \mathbb{C}^n$ for every $k = 1, \dots, n$? Similarly for \mathbb{R}^n ?

Notice that for $r = n$, this question is answered positively (Theorem 1.5) However, for $r = 1$, it is answered negatively (Proposition 1.6).

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